THE EQUIVALENCE BETWEEN CERTAIN STATISTICAL PREDICTION METHODS AND LINEARIZED DYNAMICAL METHODS

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ABSTRACT

The linearized hydrodynamic equations for storm surges are solved in analytic form for a very simple model basin and an arbitrary field of wind and pressure to show that a solution can be obtained as an integral of the product of the atmospheric forcing function and an influence function whose value tends to zero with increasing time lags. In practical cases this solution can be computed as a weighted sum of the meteorological observations during a short period before the storm surge observation.

A finite difference scheme for a slightly more general basin is then developed and the solution given formally in terms of a polynomial involving both vectors and matrices. It is shown that this solution is equivalent to the analytic solution and that both are equivalent to a linear function of the meteorological measurements of wind and pressure which must be used to obtain a description of any actual forcing function for storm surges. The technique can be generalized to provide the solution for basins of almost any shape.

The difficulties and uncertainties involved in the hydrodynamic solution are discussed, and the advantages of using a statistical method to determine the solution of the problem when sufficient data are available are shown.

1. INTRODUCTION

The rapid development of high-speed stored-program computers during the last decade has led to a rapid growth of numerical methods in meteorological research. Two distinct lines of approach have been followed. One begins with the hydrodynamic equations. These are truncated and modified as necessary to meet the requirements of mathematical and computational stability and the storage capacity of the computer being used while still retaining some of the original physical aspects of the problem. The other approach is statistical in origin and is based on the consideration of a large number of possible predictors. The weights assigned to the predictors are determined from a multiple regression program. Usually a subset containing the best or most efficient predictors is selected from the original set for practical predictions.

Of course, the two systems are not entirely independent for many of the dynamic models contain terms analogous to the Reynolds stress terms which must be evaluated empirically and some degree of physical and dynamic reasoning is usually employed in the selection of the possible predictors in the statistical models.

Both approaches have several distinct advantages. The numerical integration of the hydrodynamic equations may be continuously generalized toward a better description of nature as computers become more versatile and man acquires a better understanding of the computational process. This approach has the capability of revealing much useful information about the physical processes involved, even when the ultimate prediction is not usefully accurate. However, the result can be no better than the assumptions employed in translating the physical description of nature into mathematical terms. It can be considerably worse, for the computational process may lead to a growth or decay of energy which can be confused with hydrodynamic instability or the damping effect of friction.

The statistical approach has much less ability to reveal the underlying physical processes involved in the phenomena and has much less capability for generalization to approach a better description of nature. However, the ultimate forecast system derived, usually in the form of a regression equation or diagram, leads from the input data to the forecast by a much shorter route than that required for the dynamical approach and one can be reasonably sure of making the most efficient use of the data and theory at his disposal. Computational instabilities can develop here too, but they are much less common. The statistical program may and often does have the ability to discriminate against inferior assumptions, and sometimes makes use of implicit data, hidden correlations not clearly recognized, which serve to improve the predictions. The agreement between the results of the statistical calculations and the observations may be misleading. The data used in this type of analysis, usually time series, violate many of the basic assumptions of classical statistical theory and the results of the analysis may be more or less significant than the classical theory indicates. Tests with independent data are clearly required to establish the acceptability of any

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system derived in this manner. The sophistication of the system should not exceed that justified by the quality and quantity of the data available for development. One must remember that an accidental stratification of data can be as damaging to the resultant prediction system as a deliberate attempt to force the results.

In designing a forecast system for practical application by either method it is essential that the investigator keep in mind the quality and quantity of the input data likely to be available under operational conditions, or which can reasonably be made available through established procedures. Even a perfect computation scheme will have little value for forecasting if it requires input data that cannot be made available until after the event one wishes to predict.

2. THE STORM SURGE PROBLEM

The storm surge is defined as the difference between the actual water level during a storm and the level which would have existed in the absence of a storm. Storm surge research is generally based on the vertically integrated form of the hydrodynamic equations, usually in a linearized form. Several alternate forms of the equations have been employed, but the most significant difference between the various forms is in the degree of linearization and the convention adopted for dealing with bottom stress. Only the linearized form of the equations will be considered in this paper and thus the most general assumption that can be made about the bottom friction is to assume that it is proportional to some weighted sum of the surface stress and the mean current velocity. That is

$$\tau_b = ku^1 + \theta\tau_s \tag{1}$$

where τ_b is the bottom stress; τ_s is the surface stress; u^1 is the mean current speed along the streamline; and k and θ are constants, either or both of which may be assumed to vanish. Derivations which justify this assumption have been given by Reid [9] and Weenink [12].

The equations of motion and continuity then take the form

$$\frac{\partial U}{\partial t} - fV + gD\frac{\partial h}{\partial x} + \frac{k}{D}U = -\frac{D}{\rho_w}\frac{\partial p_a}{\partial x} + (1-\theta)\,{}^x\tau_s \tag{2}$$

$$\frac{\partial V}{\partial t} + fU + gD\frac{\partial h}{\partial y} + \frac{k}{D}V = -\frac{D}{\rho_w}\frac{\partial p_a}{\partial y} + (1-\theta)\,{}^y\tau_s \tag{3}$$

$$\frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{4}$$

where U and V are the transports along the x and y axes; f is the Coriolis parameter, $f=2\Omega \sin \phi$; ϕ is latitude; Ω is the earth's angular speed; h is the height of the free water surface above its equilibrium position; g is the acceleration of gravity; $D=D(x,y)$ is the equilibrium depth of the fluid; p_a is the atmospheric pressure; ρ_w is the density of the water; ${}^x\tau_s$ and ${}^y\tau_s$ are the surface wind stresses along the x and y axes. A derivation is given by Welander [13].

No form of the boundary condition which is rigorous from both the physical and mathematical points of view is known. It is usually assumed, however, that no fluid passes through the solid boundary when this is a coast; if the region being investigated is a bay or harbor, a portion of the boundary is fluid. This is usually treated by assuming that the height or gradient of height remains constant at the fluid boundary. Other conditions are sometimes used. There are several alternative methods of stating the initial conditions. All are equivalent to assuming that two of the four quantities, the height, the derivative of the height with time, the velocity, or the derivative of the velocity with time, are known at the initial time.

3. THE INFORMATION AVAILABLE FOR DETERMINING A SOLUTION

The form of equations (2)–(4) above imply that the pressure gradient and wind stress fields are known as continuous functions of space and time. This is never the case. Pressure observations are usually obtained in the form of total pressure at fixed points in space. Continuous observations are possible, but the data are usually readily available for computations only as point functions in time as well as in space. Information about the pressure gradient is therefore available only as a linear function of the absolute pressures at two nearby locations.

No satisfactory system for direct observations of the wind stress over water has yet been developed, nor is there any universal agreement about the proper expressions for relating the wind stress to other atmospheric variables that are directly observed. The most common assumption is that wind stress is proportional to the square of the wind speed at some standard elevation above water, but even when this convention is adopted two other troublesome details arise. Wilson [15], [16] lists the results of 47 determinations of the coefficient of proportionality as adjusted to a standard height of 10 meters. The resulting values of the drag coefficient for strong winds (about 40 kt.) vary from 1.5×10^{-3} to 4×10^{-3}. For light winds (about 10 kt.) the range is 0.4×10^{-3} to 6.2×10^{-3}. But this does not tell the whole story. Very few observations of the wind at 10 meters above the sea are available, and almost none of these was obtained during high winds. Therefore the wind velocity over water must be based on an extrapolation of values observed over land, visual estimates by sailors, or approximations based on some assumption about the relationship between the surface wind over water and the pressure gradient. In the last case one must also consider the general deficiency in pressure observations over the sea.

Although the quadratic relation between the wind speed and wind stress is more widely adopted than any other, its derivation is not entirely satisfactory. Sutton [10], [11], Dryden [2] and a few workers in this field prefer to use some exponent of the wind speed other than two (Neumann [7]). There is some empirical evidence to support the assumption of a linear relationship between the wind speed and the wind stress. This may result from some real differences between laboratory and field conditions or it may result from the fact that the relative error in wind speed squared is double that in the first power of the wind speed.

At best, the pressure or wind velocity or both are known only at a small number of discrete points, usually at discrete times, 1, 3, or 6 hours apart. The continuous or quasi-continuous specification of the pressure and wind velocity or wind stress must be obtained from this limited information by some type of interpolation. This is sometimes accomplished by fitting analytic functions to the available empirical data. Although such functions are frequently nonlinear in the space variables, they are usually linear in the observations. More often linear interpolation is used. If wind observations from land based anemometers are used, speeds must be increased somewhat to account for the decreased friction over water. If something akin to the gradient wind is used, speeds must be decreased to account for the friction that does exist at the surface. Thus after the wind speed or pressure gradient has been obtained, two empirical corrections are needed, one to adjust the observed value to that which would be expected at the standard height over water and one to convert this value of the wind speed into wind stress.

The standard mathematical treatment of time dependent differential equations requires a knowledge of the initial conditions over the whole space. Actually this is never available for the practical storm surge problem. One must "make do" with observations of the initial conditions at one or a few points near the solid boundary. Thus it is necessary to find a solution in which the error resulting from an inadequate description of the initial conditons will not increase with time.

It should be apparent from the above that any attempt to fit a solution of the hydrodynamic equations to observed data will contain a number of assumptions and approximations.

4. SOLUTION METHODS

It can be shown that any method of solving equations (2)–(4) with observed data inputs is equivalent to the construction of a regression equation expressing the storm surge at any designated location as a linear function of the initial conditions and the meteorological observations after the initial time. Moreover a practical method must not suffer too much from the lack of detail in the initial conditions and should not require the use of large volumes of data which add little skill to the predictions. One might suspect that if a best set of coefficients exists for this regression equation it must have the desired characteristics. Proving that this is so is somewhat laborious and will be attempted in three stages.

In the first stage the one-dimensional motion in a rectangular basin of constant depth will be considered. This will be used to show the general character of the solution including the tendency for the error resulting from an inadequate knowledge of the initial conditions to decrease with time. The solution will be expanded into a series of eigenfunctions. Unfortunately, it is not easy to show that the series of eigenfunctions has suitable convergence qualities. An appeal to mathematical induction will be made to show that the principal qualitative features of the solution must hold for basins of much more general character. The method used is reasonably direct and is not readily generalizable to all of the practical problems.

In the second stage a one-dimensional numerical model of much greater applicability will be developed. This approach will lead to a solution in the same form as that obtained from the first analysis but will be free of any consideration of eigenfunctions, thus eliminating the formal problem of convergence arising from the first step. The method will be readily generalizable to two dimensions and will provide some additional insight into the problems involved. Unfortunately, it is not well adapted to a display of the general character of the solution or to a proof that the error resulting from inadequate knowledge of the initial conditions tends to zero with increasing time.

Both of the above approaches lead to the same form for the regression equation. With this form well established, a statistical process of curve fitting can be used for an evaluation of the coefficients. This procedure gives less physical insight than either of the first two methods of solution, but if a sufficient amount of past data is available, it requires fewer assumptions about the physics of the problem than either of the first two methods and is much simpler from a computational point of view. Subject to the assumptions that are common to both the dynamic and statistical analysis, the statistical derivation gives assurance of maximum use of the available data. Errors may arise when the resultant solution is applied to the predictions of water levels produced by storms which differ greatly from those used in the derivation, but neither of the other two approaches eliminates this possibility. An understanding of all three derivations should minimize the possibility of an unsuspected error of this kind.

ANALYTIC SOLUTION

The equations of motion and continuity (2) and (4) suitable for a one-dimensional analysis may be written in the form:

$$\frac{\partial U}{\partial t}+gD\frac{\partial h}{\partial x}+\frac{k}{D}U=-\frac{D}{\rho_w}\frac{\partial p_a}{\partial x}+(1-\theta)\,{}^{x}\tau_s \qquad (2')$$

$$\frac{\partial h}{\partial t}+\frac{\partial U}{\partial x}=0 \qquad (4')$$

It is easy to show that equation (4′) is satisfied by defining h and v in terms of a new variable ψ, as follows

$$h=-\partial\psi/\partial x \qquad U=\partial\psi/\partial t \tag{5}$$

Substitution of (5) into (2′) gives

$$gD\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{dt^2}-\frac{k}{D}\frac{\partial\psi}{\partial t}=-\phi(x,t) \tag{6}$$

where

$$\phi(x,t)=-\frac{D}{\rho_w}\frac{\partial p_a}{\partial x}+(1-\theta)\ {}^{x}\tau_s \tag{7}$$

If the boundary conditions are taken in the form $U(0,t)=U(L,t)=0$, $\psi(x,t)$ may be expressed in the form

$$\psi(x,t)=\psi_0+\sum A_n(t)\sin\frac{n\pi x}{L} \tag{8}$$

where the $A_n(t)$ are given by

$$\frac{\partial^2 A_n}{\partial t^2}+\frac{k}{D}\frac{\partial A_n}{\partial t}+\left(\frac{n\pi}{L}\right)^2 gDA_n=B_n(t) \tag{9}$$

and

$$B_n(t)=\frac{2}{L}\int_0^L \phi(x',t)\sin\frac{n\pi x'}{L}dx' \tag{10}$$

as can be shown by the usual procedures of Fourier analysis.

Equation (9) can be recognized as a standard form of the equation for a forced harmonic oscillator with linear damping. The solution can be given in the form

$$A_n(t)=\alpha_n e^{-kt/2D}\cos\sigma_n(t-t_n)+\frac{1}{\sigma_n}\int_0^t B_n(t')e^{-k(t-t')/2D}\sin\sigma_n(t-t')dt' \tag{11}$$

$$\sigma_n^2=gD\left(\frac{n\pi}{L}\right)^2-\left(\frac{k}{2D}\right)^2$$

where α_n is the amplitude and t_n the phase of any oscillation of mode n which may have been in existence at time $t=0$. The factor exp $(-kt/2D)$, which is independent of n, in the first term on the right shows that the effect of the initial conditions will tend toward zero with increasing time. In practice one can insure that this will be true within a very short time by starting the calculation during a period in which the water level is almost constant for a prolonged time so that all α_n are small.

If the initial conditions are neglected, the important part of the solution can be obtained in the form

$$\psi=\psi_0+\sum\frac{1}{\sigma_n}\left[\int_0^t e^{-k(t-t')/2D}\sin\sigma_n(t-t')B_n(t')dt'\right]\sin\frac{n\pi x}{L} \tag{12}$$

and as ψ is defined so that $h=-\partial\psi/\partial x$, we have

$$h(t)=-\frac{\pi}{L}\sum\frac{n}{\sigma_n}\left[\int_0^t e^{-k(t-t')/2D}\sin\sigma_n(t-t')B_n(t')dt'\right]\cos\frac{n\pi x}{L} \tag{13}$$

The $B_n(t)$ defined by equation (10) may be interpreted as weighted space means of the atmospheric forcing term $\phi(x,t)$.

$$h(t)=-\frac{\pi}{L}\sum\frac{n}{\sigma_n}\left[\int_0^t W_n(t')B_n(t')dt'\right]\cos\frac{n\pi x}{L} \tag{14}$$

where

$$W_n(t-t')=e^{-k(t-t')/2D}\sin\sigma_n(t-t') \tag{15}$$

The W_n (t') may be interpreted as weighting functions which show the effect of the atmospheric forcing terms applied at time t' on the height of the free surface at time t. It can be seen that each W_n is equal to zero when t–t' equals zero, rises to a maximum, and then oscillates around the value zero with decreasing amplitude. In the derivation of equation (13) the depth and width have been assumed constant in order to obtain a separation of the space and time variables and a relatively simple form of the analytic solution. It is well known, however, that the solution of a differential equation is a continuous function of its coefficients. Therefore the actual solution, however difficult to obtain in analytic form, should differ but little from (14) if the depth and width were allowed to change slowly. That is to say, any small change in the law governing the depth and width will produce only a small change in the solution. This process can be repeated continuously. The application of this principle does not give us any quantitative information about the solution for natural basins, but it does show us that any solution to the problem must be topologically similar to a weighted sum of terms similar to those specified in equation (14). The derivation presented is given only for one space dimension, but the two-dimensional problem reduces to this when there are no significant variations in the y direction. The continuation process can be extended into two dimensions as well as into the domain of variable coefficients. This line of reasoning is essential to the subsequent development only in that it shows that the initial conditions and the force applied in the distant past cease to have any significant effect on the solution after some finite time. This could have been established more directly by an appeal to physical reasoning as showing that any attenuating factor such as friction ultimately eliminates all of the initial energy of a system.

The above line of reasoning is useful, however, when we come to evaluate the coefficients to be determined empirically in a later section, for it does give us some definite ideas about the possible shapes for the weighting functions which can arise in practical problems, at least to the point that some solutions which may arise from poorly selected empirical data can be rejected as unsuitable.

Equation (10) for the $B_n(t)$ is derived with the assumption that $\phi(x, t)$ is known as a continuous function for the entire basin being considered. This is never the case and $\phi(x,t)$ must be replaced by some sort of interpolation procedure which will supply approximations to the true value, wherever these are required, which are consistent with the known values from observing stations. Normally one would use a linear interpolation scheme and in this case the integral can be replaced by a finite sum of the type

$$B_n(t)=\sum_{\substack{j\\k}} a_{j,k,n}\phi_{j,k}(t) \tag{16}$$

where $\phi_{j,k}(t)$ is the value of $\phi(x,t)$ at station j and time t. The subscript k is used to identify the component of the forcing term. $k=1$ corresponds to the wind stress in the x direction and $k=2$ corresponds to the atmospheric pressure at station j. Pressure gradients will be determined implicitly from the pressure values at two or more stations. The expression can be extended to two dimensions by letting $k=3$ identify the wind stress along the y axis. The subscript j is used to identify the observation station. The coefficients required in (16) can be derived from the assumed interpolation formula, wind stress law, wind stress coefficient, and friction coefficients, at least for the simple case on which the analytic solution is based.

The integrals in equations (12), (13), and (14) are derived under the assumption that the $B_n(t)$ are known as continuous functions of time. In practice one is generally restricted to observations taken at discrete times, usually no less than one hour apart. Therefore it is generally necessary to interpolate between the observation times to define a continuous function. Thus the integral in equation (14) as well as that involved in equation (10) can be expressed no better than as a finite sum of the type

$$\int_0^t W_n(t-t')B_n(t')dt'=\sum_i b_{i,n}\sum_{\substack{j\\k}} a_{j,k,n}\phi_{j,k}(t-i\Delta t) \tag{17}$$

where Δt is the interval between observations, and i is the number of time intervals between the meteorological observation and its effect on the water level.

Combining equations (13), (16), and (17), we can now write the equation for the water level at any designated point as

$$h(x_0,t)=\sum_{\substack{i,j\\k}} c_{i,j,k}\phi_{j,k}(t-i\Delta t) \tag{18}$$

where

$$c_{i,j,k}=\sum_n \frac{\pi}{L}\frac{n}{\sigma_n} b_{i,n}a_{j,k,n}\cos\frac{n\pi x_0}{L} \tag{19}$$

Convergence of the sum with respect to i is assured from the form of the integral in (12). Convergence with respect to j and k is assured because there are only a finite number of observation stations to be considered and only two or three meteorological quantities to be considered at each station. Convergence with respect to n can be inferred from the form of equation (10) but cannot be rigorously established for all possible $\phi(x,t)$ by this development. The coefficients $c_{i,j,k}$ can be computed from the assumed interpolation formulae and assumed wind stress law and wind stress and friction coefficients.

NUMERICAL SOLUTION

Several methods are available for approximating equations (2) and (4) by finite differences. Perhaps the most direct and most frequently employed is presented by the set

$$\left.\begin{aligned}&\frac{U_l^{m+1}-U_l^{m-1}}{2\Delta t}+gH_l\frac{h_{l+\frac12}^m-h_{l-\frac12}^m}{\Delta x}+\frac{k}{H_l}U_l^m=F_l^m\\&\frac{h_l^{m+1}-h_l^{m-1}}{2\Delta t}+\frac{U_{l+\frac12}^m-U_{l-\frac12}^m}{\Delta x}=0\end{aligned}\right\} \tag{20}$$

where U is volume transport;

$$F_l^m=-\frac{H_l}{\rho_w}\frac{\partial p_a}{\partial x}+(1-\theta)\tau_s;\ x=l\Delta x,\ t=m\Delta t.$$

This form of the equations is readily obtained from Platzman [8] or Welander [13]. It will be assumed that proper stability criteria are maintained.

If

$$\frac{H_{l+\frac12}-H_{l-\frac12}}{H_l}<<1 \tag{21}$$

it is possible to eliminate U between these equations to obtain

$$\begin{aligned}&[h_l^{m+2}-2h_l^m+h_l^{m-2}]+4g\left(\frac{\Delta t}{\Delta x}\right)^2 H_{l+\frac12}(h_{l+1}^m-h_l^m)\\&\qquad -H_{l-\frac12}(h_l^m-h_{l-1}^m)+2k\Delta t(h_l^{m+1}-h_l^{m-1})/H_l\\&\qquad\qquad =4\left(\frac{\Delta t}{\Delta x}\right)^2[F_{l+\frac12}^m-F_{l-\frac12}^m]\end{aligned} \tag{22}$$

If $H(x)$ has continuous slope, as it does in almost all practical cases, condition (21) can always be achieved by choosing sufficiently small Δx. Thus condition (21) does not result in any formal difficulty in dealing with arbitrary depths. It may lead to serious computational difficulties if the equations are solved by conventional methods.

Since U has been eliminated in equation (22) it is necessary to restate the boundary conditions, ($U(0)=U(L)=0$), as functions of h. This can be accomplished by setting $U=0$ in equation (2′) and using the definition of F_l^m given below (20) to obtain

$$gH(0)\frac{\partial h(0)}{\partial x}=F_0^m$$

$$gH(L\Delta x)\frac{\partial h(L\Delta x)}{\partial x}=F_L^m$$

and expressing h at the boundaries by

$$h_0^m = h_1^m - \left[\frac{\partial h}{\partial x}\right]_0^m \Delta x$$

$$h_L^m = h_{L-1}^m + \left[\frac{\partial h}{\partial x}\right]_L^m \Delta x \tag{23}$$

Other expressions could be used, but these appear to be the simplest.

Equation (22) expresses h^{m+2} as a function of h for four earlier time periods. However, the first term in square brackets can be recognized as the equation obtained by approximating $\partial^2 h/\partial t^2$ by the central differences of central differences, that is by the expression

$$\frac{\partial^2 h}{\partial t^2} = \frac{h_l^{m+2} - 2h_l^m + h_l^{m-2}}{4\Delta t^2}$$

A better approximation could be expected from the more usual expression

$$\frac{\partial^2 h}{\partial t^2} = \frac{h_l^{m+1} - 2h_l^m + h_l^{m-1}}{\Delta t^2}$$

Taking advantage of this equivalence to reduce the five-term recursion formula to one involving only three time steps and collecting terms, one obtains the prediction equation

$$h_l^{m+1} = A_l h_l^{m-1} + E_l h_{l-1}^m + C_l h_l^m + G_l h_{l+1}^m + D_l {}^*F_l^m \tag{24}$$

where:

$$R_l = 4 + \frac{2k\Delta t}{H_l}$$

$$A_l = -(4 - 2k\Delta t/H_l)/R_l$$

$$E_l = -4g(\Delta t^2/\Delta x^2) H_{l-1/2}/R_l$$

$$C_l = \left[8 + 4g\left(\frac{\Delta t}{\Delta x}\right)^2 (H_{l+1/2} + H_{l-1/2})\right]/R_l$$

$$G_l = -4g(\Delta t^2/\Delta x^2) H_{l+1/2}/R_l$$

$$D_l = 4(\Delta t^2/\Delta x)/R_l$$

$${}^*F_l^m = F_{l+1/2}^m - F_{l-1/2}^m : 0 < l < L$$

$${}^*F_0^m = F_0^{m+1}$$

$${}^*F_L^m = F_L^{m+1} \tag{25}$$

An even more compact expression can be obtained by introducing the matrices A, B, and D whose elements $A_{l,p}$, $B_{l,p}$, $D_{l,p}$ are defined as follows:

$A_{0,1} = A_1$
$A_{l,l} = A_l$
$A_{L,L-1} = A_{L-1}$
$A_{l,p} = 0$ for all other values of l and p
$B_{0,0} = E_1$
$B_{0,1} = C_1$
$B_{0,2} = G_1$

$$\left.\begin{aligned} B_{l,l-1} &= E_l \\ B_{l,l} &= C_l \\ B_{l,l+1} &= G_l \end{aligned}\right\} 0 < l < L$$

$B_{L,L-2} = E_{L-1}$
$B_{L,L-1} = C_{L-1}$
$B_{L,L} = G_{L-1}$
$B_{l,p} = 0$ for all other values of l and p
$D_{0,0} = -2\Delta x/gH_0$
$D_{0,1} = D_1$
$D_{l,l} = D_l$
$D_{L,L-1} = D_{L-1}$
$D_{L,L} = 2\Delta x/gH_L$
$D_{l,p} = 0$ for all other values of l and p

This permits the prediction equation to be rewritten in the form

$$h^{m+1} = Ah^{m-1} + Bh^m + D^*F^m \tag{26}$$

where the h's and the $^*F$ are vectors, and A, B, and D are matrices. If $h(x)$ is known for $m=0$ and 1, and the meteorological variables are known, $h(x)$ for any later time period can be found by repeated application of formula (26). It has been shown above that after a sufficient period of time the effect of the initial conditions vanishes. Moreover, one usually begins a set of storm surge predictions at a time at which the initial disturbance is at a minimum. Therefore, we may disregard the initial conditions and consider only the influence of the atmosphere on a body of water which is initially at rest. A few iterations of (26) are shown below:

$$h^1 = D^*F^0$$
$$h^2 = BD^*F^0 + D^*F^1$$
$$h^3 = (A+B^2)D^*F^0 + BD^*F^1 + D^*F^2$$
$$h^4 = (AB+BA+B^3)D^*F^0 + (A+B^2)D^*F^1 + BD^*F^2 + D^*F^3 \tag{27}$$

It is seen that the vector h^{m+1} takes the form

$$h^{m+1} = M^{i*}F^{m-i} \tag{27'}$$

where M^i is a matrix polynomial generated by the repeated application of equation (26). The first few M^i as taken from (27) are

$$M^1 = D$$
$$M^2 = BD$$
$$M^3 = (A+B^2)D$$
$$M^4 = (AB+BA+B^3)D$$

If one considers the time variations of h at a single value of x, say x_0, equation (26) can be written in the form

$$h(x_0, t) = \sum_m a_m \sum_{l,k} c_{l,k} {}^*F_{l,k}(t - m\Delta t) \tag{28}$$

Equation (28) is similar in form to equation (18) except that it does not have to be summed over n. The coefficients a_m and $c_{l,k}$ can be computed from the form of the finite difference equations and the assumptions made about the wind stress and bottom friction. Summation over j, the locations from which empirical data are available, is replaced by summation over l, the mesh points in space. The number of mesh points required in the calculation is independent of the number of locations from which data are available and is usually much larger. The forcing function at the additional points must be obtained from some type of interpolation between the observing stations. Summation over i, the observation intervals, is replaced by summation over m, the computation intervals. The requirements of computational stability place an upper limit on the computational interval which is usually much smaller than the interval between observations. Again the missing empirical data must be supplied by some type of interpolation.

If we interpolate in time and space, to obtain the values of $*F(x_l,t)$ required for the numerical solution but not supplied by the observations, we obtain

$$*F_{l,k}(t-m\Delta_c t)=\sum_{\substack{i\\j}} d_{i,j,k,l,m}F_{j,k}(t-i\Delta_o t) \tag{29}$$

Subscripts c and o have been added to the Δ's in this equation to distinguish between computation and observation intervals. This distinction is not essential to any of the other equations. The coefficients can be computed from the interpolation formula used to supply the meteorological data at each mesh point and computation time from the more limited meteorological observations.

By combining equations (28) and (29) we obtain

$$h(x_0,t)=\sum e_{l,m}\sum d_{i,j,k,l,m}F_{j,k}(t-i\Delta t) \tag{30}$$

where $e_{l,m}=a_m c_{l,k}$

The interpolation process, although necessary for the numerical solution of the hydrodynamic equations, adds no new information. Therefore we may formally sum over l and m to obtain

$$h(x_0,t)=\sum f_{i,j,k}F_{j,k}(t-i\Delta t) \tag{31}$$

where

$$f_{i,j,k}=\sum_{\substack{l\\m}} e_{l,m}d_{i,j,k,l,m}$$

Equation (31) is similar in form to (19) and (28). Like (19) it is to be summed only over the observation points and observation times. Like (28) it does not have to be summed over n. However, it contains all of the data available for a determination of the solution, but none of the auxiliary functions required by the process of solution. We may therefore assume that this is the result that we would have obtained by summing equation (19) over n if we had been able to do so. Moreover, this derivation does not require any unreasonable assumptions about the bottom profile. The practical difficulties which may have attended the stipulation of mesh lengths short enough to satisfy condition (21) have now been formally eliminated. The coefficients $f_{i,j,k}$ could be computed from the hydrodynamic equations and the assumptions necessary for a numerical solution to equations (2′) and (4′). However, many of the assumptions, which must be introduced into this process, although reasonable, are in no sense unique. Their use must be justified a posteriori by comparison of the computation results with observed data. Therefore the coefficients $f_{i,j,k}$ deduced from theory and assumptions are likewise not unique, and as their ultimate justification must come from empirical evidence, one is led to consider the possibility of obtaining them directly from the empirical data.

THE REGRESSION EQUATION

With the form of the regression equation (31) well established from physical and mathematical considerations, we now consider the evaluation of the coefficients. As stated above, the theoretical derivation of the coefficients, although possible, is based on a number of unverifiable assumptions. Several sets of assumptions, whose relative validity cannot be determined a priori, appear reasonable. Consequently there is no method for determining that any set of coefficients derived from theoretical considerations will be the best possible. However, if a sufficiently long period of record is available it is possible to use the methods of linear algebra to compute the coefficients from known values of the storm surge and the meteorological variables. This procedure would be ideal if the theory were exact and a sufficient number of exact observations were available. Unfortunately, the theory is only an approximation and many of the observations are not as representative as one would like. Therefore, it is desirable to use many more equations than coefficients and to obtain a solution in a least squares sense. But this procedure is identical with the solution of a problem in multiple regression analysis, and the original hydrodynamic problem has been converted into a statistical problem.

This procedure has at least two other attractive features. The hydrodynamic equations (2) and (4) are greatly simplified and fail to recognize some physical causes for rises in sea level that are correlated with pressure and wind stress in much the same way as is the volume transport. For example, equation (2) implies that the sea level will rise in regions of low pressure because of mass transport. In many areas significant rises also occur due to heavy rainfall. This is also correlated with low pressure, and a correction for the effects of rainfall is included, although concealed, in the coefficients of the pressure term in equation (24). Evidence is accumulating that waves breaking against the coast can make a significant contribution to the total storm surge along coasts exposed to violent wave action. This phenomenon is significant only when high waves are approaching the coast; that is, for the most part only during strong onshore winds. Thus an average correction for this condition also is included in the coefficients for the wind effect in equation (24); the correlation procedure therefore makes some allowances for our lack of physical knowledge. It would be possible to eliminate these effects from the correlation procedure if exact expressions for them were available. In the meantime the implicit recognition of these factors by the coefficients derived by the statistical procedure is a useful byproduct of the procedure.

Moreover, as we have seen, the dynamic calculations require a redundancy in the input data, as the number of calculation points for which data are needed is generally much greater than the number of points for which data are available, and the time interval employed in the calculations is generally much smaller than the interval between successive observations. Any interpolation procedure used must generate some errors in the data field, and no matter how small these may be they cannot be expected to lead to any systematic improvement in the predictions.

A significant advantage of the regression method of computing storm surges is that the number of locations and time periods for which predictions are made is determined by the availability and real need for data and not by the needs of the computational system. A second advantage is that the regression equation uses only the observed data and does not require any interpolation in either time or space. This great reduction in arithmetic may make the regression technique desirable even when no past data are available for a statistical determination of the coefficients. In this case the coefficients for the regression equation can be computed by using the numerical solution of the hydrodynamic equations.

Several methods of accomplishing this could be suggested and a detailed discussion of this problem will be deferred to a later paper.

5. EXTENSION TO TWO SPACE DIMENSIONS

The extension of the analytic solution to two space dimensions is not difficult if one is willing to restrict attention to rectangular basins of constant depth, and neglect the rotation of the earth. It can be accomplished for several other basins of rather simple geometry. However, a general solution which is applicable to natural basins is not available. Even the simplest two-dimensional solution cannot be obtained from the foregoing development in a straightforward manner.

The numerical solutions can be obtained in a straightforward manner from the finite difference form of equations (2)–(4). A finite difference expression for these equations is given by Welander [13]. The extension of the statistical model to two dimensions can be accomplished simply by extending the range of the index k to include the wind stress in a direction normal to the x axis.

The algebra required for the computation of the matrix polynomials required for a proof of the validity of the method, similar to that given above, would be formidable but it could be carried out. For the present we shall satisfy ourselves by pointing out that in each stage of the calculations, $h(x,y,t+\Delta t)$ is computed as a linear function of other quantities, each of which is defined in a similar manner until ultimately every variable is a sum of the forcing functions for earlier times. Thus it should be possible in principle to compute the coefficients relating $h(x,t)$ to the sums of the available meteorological observations for earlier times.

The statistical method, in practice, is very similar to the influence method described by Welander [13] and the empirical method described by Wilson [16]. The motivation is, however, quite different, and it is believed that the foregoing derivation reveals more of the underlying physical and mathematical principles behind the method, and that an understanding of these principles should guide one to a better selection of the possible predictors. Some of the uncertainties mentioned by Wilson [16] and other uncertainties discussed by Wilson and Harris [17] should be resolvable by the derivation given above.

6. ALTERNATIVE DERIVATIONS OF THE BASIC EQUATION

Two theoretical derivations of equation (31) are given above. One may arrive at the same destination by several other routes. For example, one may regard the body of water being considered as a linear filter whose input is the time series observations of the pertinent meteorological variables and whose output is the variation of water level at any selected location bordering on or within the water body. If the problem is viewed in this way the techniques discussed by Wiener [14] are applicable. Wiener's technique was developed for phenomena in which future events are only incompletely determined by past events because of fundamental physical principles. An example of a phenomenon of this type mentioned by Wiener is the displacement of a molecule of fluid because of Brownian motion. Another problem familiar in physical oceanography is that of providing an exact description of sea state, $h(x,y,t_2)$, at time t_2 when an exact description, $h(x,y,t_1)$ is given for time t_1.

The problem we are considering should be completely deterministic if complete information about the initial conditions, forcing functions, the stress laws, and stress coefficients were known exactly, and if the linear theory were completely valid. However, a certain indeterminacy exists because of incomplete knowledge in the initial conditions, forcing functions, physical constants, and fundamental theory; and the practical situation is very much like that described by Wiener. If this paper were being prepared primarily for statisticians or communication engineers, the groups addressed by Wiener, it might be better to use the theory of time series as a starting point. Certainly that theory can contribute significantly to an extension of the results we have obtained. However, this paper is being developed for hydrodynamicists and civil engineers, and the development given above appears to the author to be the most natural.

Equation (11) can be recognized as a convolution integral involving the applied force and the eigenfunction of the basin being discussed. This fact could be used as another starting point for the development of the theory. Without doubt other approaches could be used and perhaps some of them have been. However, the author is not acquainted with any other development which shows that an exact equivalence can be developed between a multiple regression technique and the solution of a set of linear differential equations whose nonhomogeneous terms contain empirical functions.

7. COMPUTATION OF THE COEFFICIENTS BY THE LEAST SQUARES METHOD

At least one additional advantage can be sought in the statistical evaluation of the regression coefficients. The meteorological data contain a great deal of redundancy even when data are considered only at the scheduled observation times and existing weather stations. One would suspect that predictions nearly as good as the best obtainable could be derived from a regression equation which requires only a small fraction of the available data. It is desirable to take advantage of this possibility in making routine predictions, as reducing the amount of arithmetic not only reduces the amount of work and time involved, but also reduces the chance of an error in the calculations. Statisticians have been studying the problem of determining the best set of predictors from a larger set of possible predictors for several years. A systematic procedure for accomplishing this, called a screening procedure, was described by Wherry [18]. Later papers were published by Lubin and Summerfield [4]. The procedure was applied to meteorological problems by Miller [5] in an unpublished paper presented in 1956. He discussed the technique in considerable detail in a report [6] which received limited distribution in 1958. The idea received additional development and application by several other meteorologists during the next few years and two papers giving a description of the mathematical procedures involved appeared in 1959 (Aubert, Lund, and Thomasell [1]; and Klein, Lewis, and Enger [3]). The heart of the method as applied in meteorology is a high-speed computer program for computation of a set of regression equations based on an increasing number of independent variables until some cut-off point is reached.

Several versions of the technique have been used. The essential feature, common to all is that the computer is first used to compute the means, variances, and covariances of a large number of variables, some of which are regarded as "independent" but one or more of which are assumed to be linear functions of some of the others. In considering a particular dependent variable, the independent variable most highly correlated with this is determined and the regression equation determined by this variable is computed. In the next step, the second independent variable is selected in such a way that the resulting regression equation will be the best that could be obtained with the first variable already specified. Then the third variable is selected such that it will make the maximum improvement with the first two variables already selected. The process is repeated until all variables have been considered or until it is previously terminated by some programmed decision process. Variables which contribute no improvement to the system are discarded.

If the theoretical relation between the dependent and independent variables is not known, the process is usually terminated after a designated number of coefficients have been computed or when the value of the theoretical significance of the additional variables drops below some previously specified value. If the theoretical form of the regression is known from other considerations, as in this problem, it is worthwhile to consider all possible predictors, but for practical considerations one may wish to use only the most efficient predictors for practical prediction and the most complete equations only for further theoretical work.

The basic screening program has been altered to accept meteorological data in the form of hourly records of pressure, wind direction, and speed as available in the punch card decks at the National Weather Records Center; to compute the wind stress along two orthogonal axes; and to lag these in an arbitrary manner as appears to best satisfy the requirements of the above development within the memory capacity of the IBM 7090. It is practical to consider predictions for more than one location and to consider the consequences of suppressing certain data which may not be available at the time of the forecast during a single examination of the data.

A report of the application of this procedure to practical storm surge problems is in preparation for publication.

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